Soliton dynamics in periodically modulated directional couplers

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Using the variational approximation, we analyze the evolution of solitons in a twin-core nonlinear optical fiber whose coupling coefficient has both constant and spatially periodic parts. The influence on switching of direct and parametric resonances between the period of the energy oscillation within the coupler and the periodic modulation is considered, as is the influence of the modulation on a bifurcation that splits the stationary symmetric soliton into a pair of asymmetric ones.

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I. INTRODUCTION

Using solitons for switching in nonlinear directional couplers has attracted much attention and inspired many theoretical investigations [1–3]. In most cases, the examined coupler consists of two identical, parallel waveguides, so that the coupling coefficient is constant. However, there is significant experimental interest in couplers which do not keep the coupling coefficient constant; for example, grating assisted couplers [4]. Couplers with varying separation between the two cores or periodic modulation of the cores' refractive indices are both readily fabricated, and such a spatial variation of the coupling coefficient may create more possibilities for coupling and switching régimes [5].

Since a soliton in the homogeneous coupler is apt to produce a periodic oscillation of energy between the cores, it is natural to analyze the soliton's dynamics in a coupler, periodically modulated in space. Interference between this modulation and the proper soliton's oscillations may cause a resonance, thereby producing new dynamical régimes in the coupler. Resonance between small periodic modulations and the internal oscillations of a bound state of two solitons in weakly coupled fibers was considered in Ref. [6]. In contrast, here we consider a single soliton launched into one arm of the coupler, and examine its evolution.

The soliton oscillations resonantly driven by the periodic modulation are modeled with coupled cubic Schrödinger equations describing the envelopes of the electromagnetic waves in the two waveguides:

$$\begin{aligned} iu_z+\tfrac{1}{2}u_{\tau\tau}|u|^2u &= -[1+\epsilon\cos(kz)]v \ , \\ iv_z+\tfrac{1}{2}v_{\tau\tau}+|v|^2v &= -[+\epsilon\cos(kz)]u \ , \end{aligned} \tag{1}$$

where the usual scaling of variables has been done, the constant part of the coupling coefficient has been normalized to unity, and ϵ and k are the amplitude and wave number of the modulation, respectively. In all cases we take $\epsilon<1$, so that the full coupling coefficient does not change sign. Although numerical solutions always start from an initial condition where all the energy is confined to one core, analysis of the stability does consider the case of some energy initially being in each arm of the coupler. We are interested in values of the modulation's wave number k which are matched in some way to this initial energy. Either k equals the wave number of the soliton's oscillation, or is twice that value. These two cases correspond, respectively, to direct and parametric resonance at small values of ϵ.

The analysis will develop in the semianalytical approximation based on the Lagrangian formalism associated with Eq. (1). Following this approximation [2,3], one assumes the temporal and spatial variations of the soliton to factor as

$$u=\eta\,\mathrm{sech}(\eta\tau)\cos(\theta(z))e^{i\phi(z)+i\psi(z)} \ ,$$
$$v=\eta\,\mathrm{sech}(\eta\tau)\sin(\theta(z))e^{i\phi(z)-i\psi(z)} \ .$$

Substituting this into the Lagrangian density corresponding to Eq. (1), integrating over τ, and varying the resulting effective Lagrangian lead to equations for the evolution of $\theta(z)$ and $\psi(z)$. A straightforward generalization of those in Refs. [2] and [3], these are

$$\begin{aligned} \frac{d\theta}{dz} &= -[1+\epsilon\cos(kz)]\sin(2\psi) \ , \\ \frac{d\psi}{dz} &= 2P_0\cos(2\theta)-[1+\epsilon\cos(kz)]\cos(2\psi)\cot(2\theta) \ , \end{aligned} \tag{2}$$

where $P_0=\eta^2/6$ corresponds to the soliton's energy. The factor 6 normalizes this correspondence so that the switching value—the transition between full and partial exchange of energy between the two cores—is given by $P_0=1$ in the unperturbed coupler [2]. The wave number of the soliton oscillation k_0 depends on P_0 [3]:

$$k_0(P_0)=\begin{cases} \pi/K(P_0^2) & \text{if } P_0<1 \\ 2\pi P_0/K(1/P_0^2) & \text{if } P_0>1 \ , \end{cases}$$

where K is the complete elliptic integral of the first kind. An additional equation exists for $\phi(z)$, but, being independent of ψ and θ, is of no interest in what follows.

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The initial condition nominated above corresponds to $\theta(0)=0$.

It is necessary to emphasize that the soliton ansatz postulated above (in the same form as in Refs. [2] and [3]) neglects the chirp, as well as any possible variation of the width of the soliton. Recently, it has been demonstrated [7] that incorporation of the corresponding degree of freedom into the variational formalism significantly improves the accuracy of the results. However, the corresponding system of ordinary differential equations (ODEs) becomes complicated, and cannot be solved analytically, unlike the system given by Eq. (2), even in the case $\epsilon=0$. Hence here, where the problem of energy oscillations between two cores with the periodically modulated coupling constant is considered, we employ the simplest, chirpless, constant-width ansatz for the soliton's form.

In Sec. II, results are presented for the numerical solution of Eq. (2). These solutions reveal the onset of what appears to be dynamical chaos at some critical value ϵ_{crit}, which depends on the soliton's energy P_0. The further P_0 is from the critical value $P_0=1$, the larger the value of ϵ_{crit}, i.e., $\epsilon_{\text{crit}}(P_0)$ has a sharp minimum at $P_0=1$, the switching value of energy in the case $\epsilon=0$ (see Fig. 1). We also find the shifted switching energy as a function of ϵ.

In Sec. III, we present a fully analytical description of some special dynamical régimes driven by the periodic perturbation. The first corresponds, in the case $\epsilon\rightarrow 0$, to small-amplitude oscillations of the variable θ about the value 0. It is demonstrated that the parametric resonance is analytically tractable in this case, and reduces simply to a shift in the spatial wave number of the oscillations. Next we consider the effect of the periodic modulation on the recently analyzed bifurcation of the solitons [3]. This bifurcation occurs at $P_0=\frac{1}{2}$. For $P_0>\frac{1}{2}$, the stationary soliton solution $\theta(z)\equiv\pi/4$ (i.e., a symmetric distribution of energy between the two cores) becomes unstable, splitting into two mutually symmetric stable stationary solitons with unequal energies in the two cores. We show that periodic modulation shifts the bifurcation point to a smaller value of P_0, and renders the solitons produced by the bifurcation nonstationary (i.e., there are oscillations between the cores).

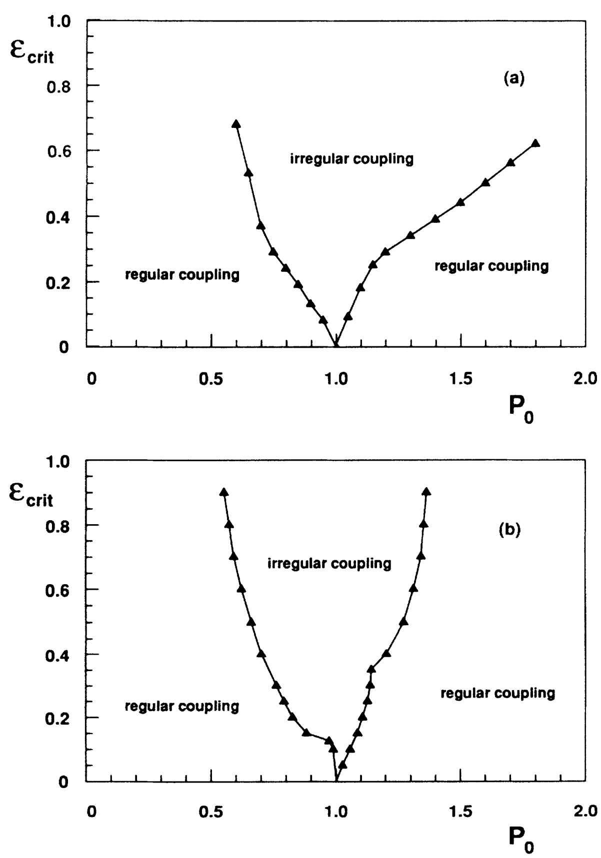

FIG. 1. The numerically determined boundary between regular and irregular behaviors. Case (a) is direct resonance; case (b) is parametric resonance.

II. DIRECT AND PARAMETRIC RESONANCES: NUMERICAL RESULTS

In this section we numerically investigate how the periodic variation of the coupling coefficient affects the coupling and switching behavior of the optical solitons. In particular, we concentrate on the direct and parametric resonances.

We study only the qualitative properties of the variation. It has been shown [8] that the variational approximation may have an error less than or of the order of 10%. Thus this approximation is adequate for this purpose. Consequently, rather than the partial differential equations of Eq. (1), we start with the ODEs of Eq. (2), which are much easier to solve with appropriate initial conditions.

From numerical investigation, it is found that the periodic perturbation of coupling can lead to chaotic (i.e., irregular) coupling of solitons in a nonlinear coupler as well as the normal (regular) coupling seen in the unperturbed coupler. Figure 1 shows the numerically determined demarcation between normal and chaotic coupling. Here we use as an operational definition of chaotic coupling that irregularity occurs throughout a coupling distance of $z=4\pi$. In Fig. 1(a), the periodic perturbation satisfies the condition for direct resonance: the spatial wave number k equals the coupling wave number $k_0(P_0)$ of the soliton in the unperturbed coupler, whereas Fig. 1(b) shows parametric resonance, i.e., $k=2k_0(P_0)$. Interestingly, in both cases, the boundary between normal and chaotic couplings caused by the perturbation has a very deep minimum at $P_0=1$, which is the critical switching threshold energy for the soliton in the unperturbed coupler. This is quite natural, since the coupling behavior at the critical point is most sensitive to any disturbance of the coupling coefficient. With this in mind we understand why the boundaries occur at higher values of ϵ, as the launched energy deviates from $P_0=1$. The behaviors—shown in Figs. 1(a) and 1(b)—for periodic perturbations of very different spatial wave number are similar. Based on this, we conclude that the coupling or switching of solitons at or near the critical energy is very sensitive to periodic perturbation of the coupling coefficient, and, as the energy changes more from this

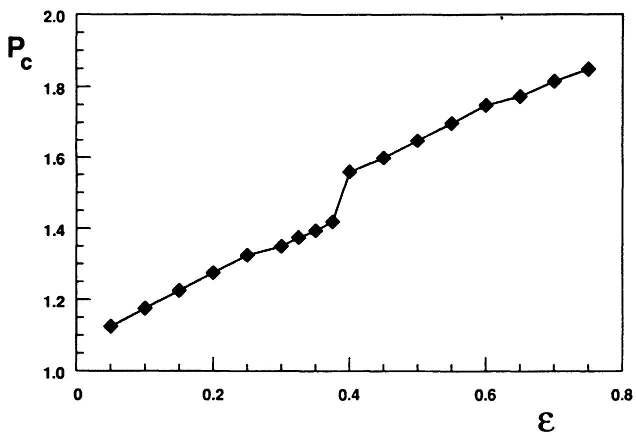

FIG. 2. The critical switching energy of the soliton as a function of the modulation amplitude ϵ for different spatial frequencies of the modulation.

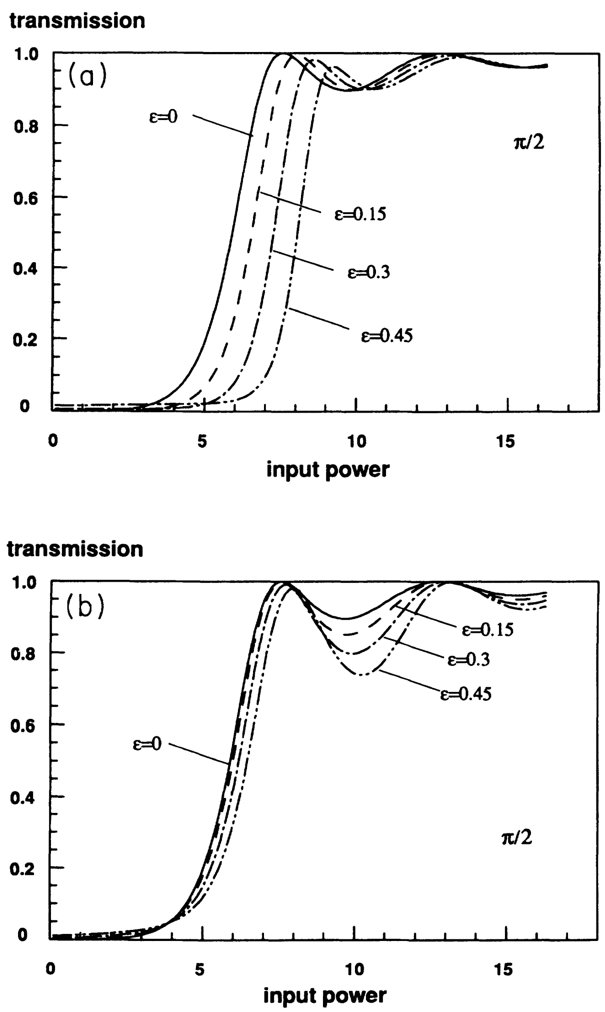

FIG. 3. Typical switching curves for the $\pi/2$ coupler with periodically modulated coupling. Different amplitudes of the modulation are shown. Cases (a) and (b) are direct and parametric resonances, respectively. The modulation period is that of the soliton corresponding to an energy $P_0=\frac{1}{2}$.

value, the system becomes less sensitive. If we define a threshold amplitude of the periodic perturbation at which chaotic coupling starts, then this threshold will vanish at the critical energy, and increase approximately in proportion to the difference between the actual energy and the critical energy.

Figure 2 shows the critical switching energy increases as the magnitude of the perturbation increases. Here the perturbation's wave number is given by $k=k_0(P_0=\frac{1}{2})$, which equals the spatial wave number of the soliton coupling in the unperturbed coupler with initial energy $P_0=\frac{1}{2}$. Choosing other valuers of k changed only the details of the curve, not the qualitative trend.

The effect of periodic perturbation on the switching performance of the coupler is illustrated in Fig. 3. These results are for the $\pi/2$ coupler. (Results for the π coupler are similar.) Here $k=k_0(P_0=\frac{1}{2})$ is used for both cases, i.e., $k=k_0(\frac{1}{2})$ for direct and $k=2k_0(\frac{1}{2})$ for parametric resonances. It is apparent that the perturbation makes slight difference to the switching behavior, except over a limited range of energies. While the switching energy is sometimes influenced significantly for the case of direct resonance, for parametric resonance it is not. Further investigation found that the range of energy over which there is an appreciable influence of the perturbation is limited and solely determined by the value of k: higher values of k lead to higher values of the energies bounding this influenced range.

III. ANALYTICAL TREATMENT

A. Influence of parametric resonance on small-amplitude oscillations

Fully analytical consideration of the parametric resonance is possible when the variable $\theta(z)$ performs small-amplitude oscillations about 0, i.e., the share of the soliton's energy in the second core remains small. This occurs when the energy of the soliton is significantly higher than the critical switching value, in which case $k_0 \rightarrow 4P_0$. In this case, Eq. (2) can be linearized in θ:

$$\frac{d\theta}{dz}=-\sin(2\psi)-\epsilon\cos(kz)\sin(2\psi)\ ,$$
$$\theta\frac{d\psi}{dz}=2P_0\theta-\tfrac{1}{2}\cos(2\psi)-\tfrac{1}{2}\epsilon\cos(kz)\cos(2\psi)\ . \tag{3}$$

In the case of parametric resonance, the fundamental wave number is $\frac{1}{2}k$. Accordingly, a resonant solution to Eq. (3) is sought in the form

$$\theta(z)=B\cos(\tfrac{1}{2}kz+\delta)\ ,$$
$$2\psi(z)=\tfrac{1}{2}kz+\phi\ , \tag{4}$$

with constants B, δ, and ϕ. Substitution of Eq. (4) into Eq. (3) leads to two solutions. One has $\delta=\pi/2=\phi$, $B=1/(2P_0)$, and

$$k=4P_0(1+\tfrac{1}{2}\epsilon)\ ; \tag{5}$$

for the other, $\delta=0=\phi$, $B=1/(2P_0)$, and

$$k = 4P_0(1 - \tfrac{1}{2}\epsilon) \ . \tag{6}$$

In this case, the parametric resonance simply shifts the spatial wave number of the soliton's small oscillations, according to Eqs. (5) and (6), without altering their amplitude.

B. Influence of periodic modulation on soliton bifurcation

Equation (2) always possesses the obvious solution $\theta(z) \equiv \pi/4$ and $\sin\psi(z) \equiv 0$, corresponding to a stationary symmetric soliton with the energy equally divided between the two cores. In Ref. [3], it was demonstrated that the symmetric soliton is unstable at $P_0 \geq \frac{1}{2}$: at the point $P_0 = \frac{1}{2}$, undergoing a so-called "pitchfork bifurcation," splitting into two stable, nonsymmetric solitons, which, however, are symmetric with respect to each other. This bifurcation was predicted in Ref. [3] on the basis of the above-mentioned constant-width ansatz for the soliton's wave form. Later [7,9] it was shown that allowing a changing width for the soliton slightly shifts the bifurcation point (the relative shift is a few percent), and it renders the bifurcating solutions unstable in a narrow region, the width of which is of the same order of magnitude. In this work, we consider the influence of the periodic modulation on the bifurcation, neglecting those relatively small corrections.

To this end, expand Eq. (2) near the point $P_0 = \frac{1}{2}$, assuming that $\epsilon, (P_0 - \frac{1}{2}), [\theta - (\pi/4)] \ll 1$. The resulting equation can be recast as one for $\theta_1(z) = \theta(z) - (\pi/4)$:

$$\frac{d^2\theta_1}{dz^2} - 8q_0^2\theta_1 + 8\theta_1^3 + 4\epsilon\cos(kz)\theta_1 = 0 \ , \tag{7}$$

with $q_0^2 = (P_0 - \frac{1}{2})$.

At $\epsilon = 0$, the above-mentioned bifurcation is included in Eq. (7): the solution $\theta_1(z) \equiv 0$ corresponds to the symmetric soliton losing its stability when q_0^2 becomes positive, i.e., at $P_0 > \frac{1}{2}$. The nonsymmetric solitons correspond to the constant solutions

$$\theta_1 = \pm q_0 \equiv \pm\sqrt{P_0 - \tfrac{1}{2}} \ . \tag{8}$$

This remains true as long as $(P_0 - \frac{1}{2}) \ll \frac{1}{2}$. It is straightforward to find spatial eigenfrequencies of infinitesimal oscillations around these stable solutions. At $P_0 < \frac{1}{2}$ (i.e., $q_0^2 < 0$), the corresponding squared eigenfrequency for the stable solution $\theta_1(z) \equiv 0$ is $-8q_0^2$, which is positive. At $P_0 > \frac{1}{2}$ (i.e., $q_0^2 > 0$), the squared eigenfrequency of oscillation about the solution given by Eq. (8) is $16q_0^2$, which also is positive.

Proceeding to nonzero values of ϵ in Eq. (7), notice that the parametric resonance introduces an instability to the solution $\theta_1 = 0$ in the small region around the point $q_0^2 = -k^2/32$ given by

$$|\tfrac{1}{32}k^2 + q_0^2| < \tfrac{1}{4}\epsilon \ . \tag{9}$$

Very close to the left edge of this region, i.e., q_0^2 negative and slightly greater than $-\frac{1}{32}k^2 - \frac{1}{4}\epsilon$, the nontrivial stable solutions produced by instability of the trivial one $\theta_1(z) \equiv 0$ are found to be

$$\theta_1(z) = \pm\frac{2}{\sqrt{3}}\sqrt{q_0^2 + \tfrac{1}{4}\epsilon + \tfrac{1}{32}k^2}\sin(\tfrac{1}{2}kz) \ . \tag{10}$$

Subsequent analysis strongly depends on the value of the modulation wave number k. If it is sufficiently large, the region defined by Eq. (9) is separated from the bifurcation point $q_0^2 = 0$, thereby leaving the bifurcation unaffected by the modulation.

However, the situation differs if $k^2 \leq 8\epsilon$, because the point $q_0^2 = 0$ is inside the region defined by Eq. (9). Here we concentrate on the limiting case of long-wave modulation, i.e., $k^2 \ll 8\epsilon$. (For the intermediate case of $k^2 \sim 8\epsilon$, the bifurcation phenomena for the system considered are very involved, and will not be considered here.) Under this condition, except in a small neighborhood of the edge point $q_0^2 \approx -\frac{1}{4}\epsilon - \frac{1}{32}k^2$, one can neglect the first term in Eq. (7) to obtain the simplest approximation:

$$\theta_1(z) = \begin{cases} 0 \ , & \text{if } q_0^2 < \dfrac{\epsilon}{2}\cos(kz) \\ \pm\left[q_0^2 - \dfrac{\epsilon}{2}\cos(kz)\right]^{1/2} & \text{if } q_0^2 > \dfrac{\epsilon}{2}\cos(kz) \ . \end{cases} \tag{11}$$

More correctly, this solution would be modified to smooth it near the points where its derivative is discontinuous, and the extended domains where it vanishes would become domains where the solution nearly vanishes. To understand a global structure of the solution [which actually means the choice of sign in Eq. (11)], we note that it must be obtained by a continuous deformation of the solution given by Eq. (10). Hence, as long as $q_0^2 < \epsilon/2$, the solution must have the form shown in Fig. 4. However, for $q_0^2 > \epsilon/2$, no longer do regions exist where the solution nearly vanishes [recall that there are regions where it vanishes in the approximation given by Eq. (11)], so the solution may have only the form shown in Fig. 5. Thus there must be an additional bifurcation

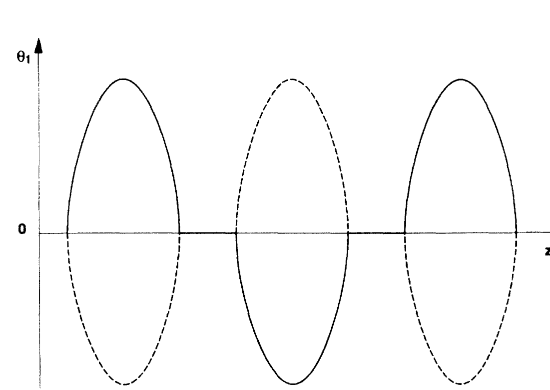

FIG. 4. The shape of the solution prescribed by Eq. (11) at $-\epsilon/4 < q_0^2 < \epsilon/2$. The solid and broken curves represent the two mutually symmetric solutions, corresponding to the two possible choices of sign.

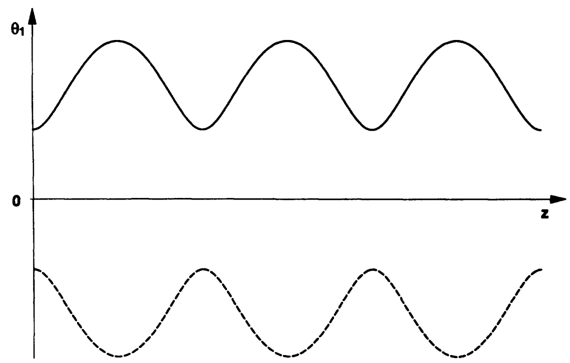

FIG. 5. The shape of the solution prescribed by Eq. (11) at $q_0^2 > \epsilon/2$. The solid and broken curves represent the two mutually symmetric solutions, corresponding to the two possible choices of sign.

where these two forms of solution separate, i.e., where the value of $\theta_1(z)$ no longer oscillates about 0, but about a nonzero average. Although we did not consider this additional bifurcation in the details, it is plausible that it occurs when the curves shown in Figs. 4 and 5 are tangential to the axis $\theta_1=0$, i.e., at $q_0^2=\epsilon/2$. That the envelopes of the oscillations defined by Eqs. (10) and (11) intersect at $q_0^2=\epsilon/2$ is also suggestive of this point.

Thus we conclude that the long-wave modulation shifts the transition between the trivial symmetric solution $[\theta(z)\equiv\pi/4]$ and a nontrivial one from the critical point $P_0=\frac{1}{2}$ to that

$$P_0=\tfrac{1}{2}-\tfrac{1}{4}\epsilon-\tfrac{1}{32}k^2\ . \tag{12}$$

Unlike the earlier findings for the case $\epsilon=0$ [3], the nontrivial solution produced by the shifted bifurcation remains symmetric. However, it is nonstationary, i.e., it oscillates between the two cores. With further increase in the energy, the additional symmetry-breaking bifurcation occurs at $P_0=\frac{1}{2}+\frac{1}{2}\epsilon$ [omitting a possible correction of order k^2; see Eq. (12)]. Finally, with yet more increase of the soliton's energy, the asymmetric solution described by Eq. (11) gradually approaches the solution given by Eq. (10), existing at $\epsilon=0$. The analytical results presented and explained above are summarized in the stylized bifurcation diagram of Fig. 6.

IV. CONCLUSION

The effect of a small amplitude, periodic perturbation on a soliton launched into one arm of a directional coupler has been examined. We found that as the energy carried by the soliton approaches the threshold energy for switching—from either above or below—the size of the perturbation required to cause chaotic coupling between the arms reduces, until, exactly at threshold, i.e., $P_0=1$, any perturbation causes chaos. Using numerical investigation, the effect of the perturbation on the switching curve was found to be minimal.

Further, analysis within the variational approximation showed how the perturbation changed the internal structure of the natural solitons of the system, and caused them to manifest internal oscillations. These oscillations were shown qualitatively to be most significant in the neighborhood of the solitons' bifurcation point, where the trivial soliton splits into nontrivial asymmetric structures. While in the unperturbed coupler these solitons are stationary, nonzero values of ϵ cause oscillation of the energy between the two cores in the neighborhood of $P_0=\frac{1}{2}$.

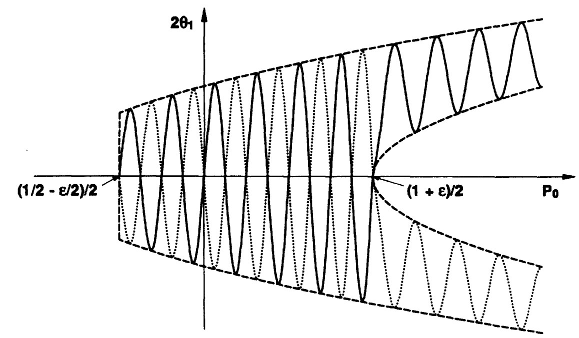

FIG. 6. A qualitative bifurcation diagram for the soliton of energy approximately $P_0=\frac{1}{2}$, in the coupler with long-wave modulation of the coupling coefficient. The solid and dotted curves represent the existence of the two solutions, corresponding to the two options for sign in Eq. (11), and describing oscillation of the energy between the two arms of the coupler. The oscillations with $-\frac{1}{4}\epsilon<P_0-\frac{1}{2}<\frac{1}{2}\epsilon$ represent the existence of symmetric solutions; the diverging curves with damped oscillations for $P_0-\frac{1}{2}>\frac{1}{2}\epsilon$ indicate the asymmetric solutions, $(P_0\rightarrow\infty)$ coinciding asymptotically with the stationary asymmetric solutions, given in Ref. [3]. The broken curves form the envelope of the soliton oscillations described by Eq. (11).

ACKNOWLEDGMENTS

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